INFILTRATION ON A STRIP IN THE PRESENCE OF AN INCLINED IMPERMEABLE HORIZON

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The author examines the effect of infiltration, acting on a strip, on the ground water level for unbounded and semibounded one-dimensional flow models, when the soil is homogeneous and the impermeable horizon has a slight tilt.

An unbounded uniform flow with normal depth h_0, on which from a certain instant of time t = 0 infiltration of intensity ε_0, acting on a strip perpendicular to the direction of flow is superimposed, is considered in relation to a homogeneous soil with a small inclination of the impermeable horizon i. The problem is to determine the ground water head created by this infiltration. Assuming that the flow is one-dimensional we can write the Boussinesq equation for the head H [1].

$$\frac{\partial H}{\partial t}=\frac{k}{\mu}\frac{\partial}{\partial x}\left[(H+ix)\frac{\partial H}{\partial x}\right]+\frac{\varepsilon(x)}{\mu}$$
$$\varepsilon(x)=\varepsilon_0=\text{const}\quad(x_1\leqslant x\leqslant x_2)$$
$$\varepsilon(x)=0\qquad(x<x_1,\ x>x_2)\tag{1}$$

Here, k is the permeability, μ the free porisity, and x the horizontal coordinate.

The problem of one-dimensional percolation without a pressure gradient in the presence of an inclined impermeable horizon was first studied by P. Ya. Polubarinova-Kochina [1]. As distinct from the problems considered in [1], the present study takes into account infiltration on a strip and adopts a somewhat different approach to the linearization of Eq. (1).

The relation between the head and the flow depth h is expressed as

$$h=H+ix$$

Going over from H to h in Eq. (1) with subsequent linearization leads to the equation

$$\frac{\partial h}{\partial t}=\frac{kh'}{\mu}\frac{\partial^2 h}{\partial x^2}-\frac{ik}{\mu}\frac{\partial h}{\partial x}+\frac{\varepsilon(x)}{\mu}$$

(h' is the averaged flow depth) which after the substitutions

$$\tau=\frac{kh'}{\mu}t,\quad 2\alpha=\frac{i}{h'},\quad w=\frac{\varepsilon(x)}{kh'},\quad w_0=\frac{\varepsilon_0}{kh'}$$
$$u(x,\tau)=[h(x,\tau)-h_0]\exp(-\alpha x+\alpha^2\tau)\tag{2}$$

takes the form

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}+w\exp(-\alpha x+\alpha^2\tau)\tag{3}$$

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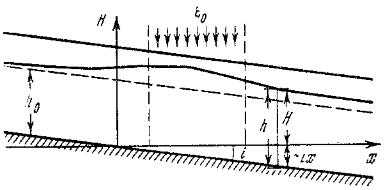

Fig. 1

We find the solution of Eq. (3) satisfying the initial condition $u(x,0) = 0$. The form of this solution in quadratures is as follows [2]:

$$u(x,\tau)=\int_0^{\tau}\int_{x_1}^{x_2}\frac{w_0\exp(-\alpha\xi+\alpha^2\tau_1)}{2\sqrt{\pi(\tau-\tau_1)}}\exp\frac{-(\xi-x)^2}{4(\tau-\tau_1)}\,d\xi\,d\tau_1$$

or what amounts to the same thing,

$$u(x,\tau)=\frac{w_0}{2\sqrt{\pi}}\exp(-\alpha x+\alpha^2\tau)\int_0^{\tau}\frac{d\tau_1}{\sqrt{\tau-\tau_1}}\int_{x_1}^{x_2}\exp\frac{-[\xi-x+2\alpha(\tau-\tau_1)]^2}{4(\tau-\tau_1)}\,d\xi$$

$$=\frac{w_0}{2}\exp(-\alpha x+\alpha^2\tau)\int_0^{\tau}\left[\operatorname{erf}\frac{x_2-x+2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}}-\operatorname{erf}\frac{x_1-x+2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}}\right]d\tau_1 \qquad (4)$$

We write the results of evaluating the integral

$$I=\int_0^{\tau}\operatorname{erf}\frac{x_i-x+2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}}\,d\tau_1$$

as follows:

$$\begin{aligned}&\text{at}\quad x_i-x>0\\ &\qquad I=I_{1,i}=\tau-u_i\operatorname{erfc}p_i-a\exp(-v_i)\operatorname{erfc}p_i^*+b\exp(-p_i^2)\\ &\text{at}\quad x_i-x<0\\ &\qquad I=I_{2,i}=-\tau+u_i\operatorname{erfc}(-p_i)+a\exp(-v_i)\operatorname{erfc}(-p_i^*)+b\exp(-p_i^2)\end{aligned} \qquad (5)$$

Here, we have introduced the notation:

$$u_i=\frac{h'}{i^2}\left[\frac{ki^2t}{\mu}-h'+i(x_i-x)\right]$$

$$p_i=\frac{1}{K\sqrt{t}}\left(x_i-x+\frac{ikt}{\mu}\right),\quad p_i^*=\frac{1}{K\sqrt{t}}\left(x_i-x-\frac{ikt}{\mu}\right)$$

$$a=\left(\frac{h'}{i}\right)^2,\quad v_i=\frac{i(x_i-x)}{h'},\quad b=\frac{Kh'\sqrt{t}}{i\sqrt{\pi}}$$

$$K=2\left(\frac{kh'}{\mu}\right)^{1/2}$$

Now, using Eqs. (2), (4), and (5), we determine the ground water head $\Delta h = h - h_0$ on different intervals of variation of x:

$$\begin{aligned}x\leqslant x_1,&\quad \Delta h={}^1/_2w_0(I_{1,2}-I_{1,1})\\ x_1\leqslant x\leqslant x_2,&\quad \Delta h={}^1/_2w_0(I_{1,2}-I_{2,1})\\ x>x_2,&\quad \Delta h={}^1/_2w_0(I_{2,2}-I_{2,1})\end{aligned} \qquad (6)$$

Letting t tend to infinity, we note that the depression curve tends to a certain limiting position everywhere except for a region infinitely remote downstream ($x \to \infty$), where the nonsteady process continues.

As $t \to \infty$, Eqs. (6) take the form

$$\begin{aligned}x\leqslant x_1,&\quad \Delta h=\frac{\varepsilon_0h'}{ki^2}\left[\exp\frac{-i(x_1-x)}{h'}-\exp\frac{-i(x_2-x)}{h'}\right]\\ x_1\leqslant x\leqslant x_2,&\quad \Delta h=\frac{\varepsilon_0h'}{ki^2}\left[1-\exp\frac{-i(x_2-x)}{h'}+\frac{i}{h'}(x-x_1)\right]\\ x\geqslant x_2,&\quad \Delta h=\frac{\varepsilon_0}{2ki}(x_2-x_1)\operatorname{erfc}\frac{x-2\alpha\tau}{2\sqrt{\tau}}\end{aligned} \qquad (7)$$

If in (7) $x \ll 2\alpha\tau$, then

$$\operatorname{erfc}\frac{x-2\alpha\tau}{2\sqrt{\tau}}\approx 2,\qquad \Delta h=\frac{\varepsilon_0}{ki}(x_2-x_1)$$

At these values of x, the flow must be

$$Q = -k\frac{\partial H}{\partial x}h = kih_0 + \varepsilon_0(x_2 - x_1).$$

As compared with the original flow, it has changed by an amount equal to the total infiltration $\varepsilon_0(x_2 - x_1)$.

We introduce the new variable

$$\xi = \frac{x - 2\alpha\tau}{2\sqrt{\tau}}.$$

Then Eq. (7) takes the form

$$\Delta h = \frac{\varepsilon_0(x_2 - x_1)}{2ki}\operatorname{erfc}\xi$$

and ξ may take any values. Setting $\xi = c$, we obtain a system

$$\frac{x - 2\alpha\tau}{2\sqrt{\tau}} = C$$

which represents sections moving along the x axis carrying constant values of the flow depth. Geometrically, these constitute a family of curves in the (x,τ) plane with the property that along each of them a constant value of the function $h(x,\tau)$ is preserved. It is easy to determine the velocity of these sections:

$$v = \frac{dx}{dt} = \frac{ik}{\mu} + c\left(\frac{kh'}{\mu t}\right)^{1/2} \approx \frac{ik}{\mu}$$

We will now consider the ground water head in a semibounded bed with infiltration on a strip.

In this case, the solution of Eq. (1) must clearly satisfy not only the initial but also the boundary condition, which we take in the following form: $h(0,t) = h_0$.

However, on going over to the function $u(x,\tau)$ we have Eq. (3), which may be written in the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \delta(x,\tau)$$

and the two conditions

$$u(x,\ 0) = 0 \qquad (8)$$

$$u(0,\ \tau) = 0 \qquad (9)$$

We continue the region of definition of $u(x,\tau)$ onto the other x semiaxis, assuming that condition (8) is satisfied there too. In this auxiliary region of definition we select $\delta(x,\tau)$ so that condition (9) is satisfied. We achieve this by setting

$$\delta(x,\tau) = \begin{cases} w_0\exp(-\alpha x + \alpha^2\tau), & x_1 \leqslant x \leqslant x_2 \\ -w_0\exp(\alpha x + \alpha^2\tau), & -x_2 \leqslant x \leqslant -x_1 \\ 0,\ x \notin [x_1, x_2], & x \notin [-x_2, -x_1] \end{cases}$$

Thus, the problem is reduced to the study of the flow in an unbounded bed, and in the region in question the unbounded and semibounded flows coincide.

In integral form the solution of the problem is written

$$u(x,\tau) = \int_0^\tau\left[\int_{x_1}^{x_2}\frac{w_0\exp(-\alpha\xi + \alpha^2\tau_1)}{2\sqrt{\pi(\tau-\tau_1)}}\exp\frac{-(\xi - x)^2}{4(\tau-\tau_1)}d\xi - \int_{-x_2}^{-x_1}\frac{w_0\exp(\alpha\xi + \alpha^2\tau)}{2\sqrt{\pi(\tau-\tau_1)}}\exp\frac{-(\xi-x)^2}{4(\tau-\tau_1)}d\xi\right]d\tau_1$$

$$= \frac{1}{2}w_0e^{\alpha^2\tau}\left\{e^{-\alpha x}\int_0^\tau\left[\operatorname{erf}\frac{x_2 - x + 2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}} - \operatorname{erf}\frac{x_1 - x + 2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}}\right]d\tau_1\right.$$

$$\left.+ e^{\alpha x}\int_0^\tau\left[\operatorname{erf}\frac{-x_2 - x - 2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}} - \operatorname{erf}\frac{-x_1 - x - 2\alpha(\tau-\tau_1)}{2\sqrt{\tau-\tau_1}}\right]d\tau_1\right\}$$

As a result of evaluating the integral

$$J = \int_0^{\tau} \operatorname{erf} \frac{x_i - x - 2\alpha(\tau - \tau_1)}{2\sqrt{\tau - \tau_1}} d\tau_1$$

we obtain

at $x_i - x \geqslant 0$

$$J = J_{1,i} = \tau - u_i^* \operatorname{erfc} p_i^* - a \exp v_i \operatorname{erfc} p_i - b \exp(-p_i^{*2})$$

at $x_i - x \leqslant 0$

$$J = J_{2,i} = -\tau + u_i^* \operatorname{erfc}(-p_i^*) + a \exp v_i \operatorname{erfc}(-p_i) - b \exp(-p_i^{*2})$$

Here

$$u_i^* = \frac{h'}{i^2}\left[\frac{ki^2 t}{\mu} - h' - i(\tau_i - x)\right]$$

We now write out the equations for Δh. There are two possible cases: a) the flow is bounded above ($0 \le x \le \infty$); b) the flow is bounded below ($-\infty < x \le 0$):

a) $x_1, x_2 > 0,\ x_3 = -x_1 < 0,\ x_4 = -x_2 < 0$

$$\begin{aligned} &0 \leqslant x \leqslant x_1, \quad \Delta h = {}^1/_2 w_0[I_{1,2} - I_{1,1} + \exp(2\alpha x)(J_{2,4} - J_{2,3})] \\ &x_1 \leqslant x \leqslant x_2, \quad \Delta h = {}^1/_2 w_0[I_{1,2} - I_{2,1} + \exp(2\alpha x)(J_{2,4} - J_{2,3})] \\ &x \geqslant x_2, \quad \Delta h = {}^1/_2 w_0[I_{2,2} - I_{2,1} + \exp(2\alpha x)(J_{2,4} - J_{2,3})] \end{aligned} \tag{10}$$

As $t \to \infty$, Eqs. (10) may be rewritten as follows: $0 \le x \le x_1$

$$\Delta h = \frac{\varepsilon_0 h'}{ki^2}\left(\exp\frac{ix}{h'} - 1\right)\left(\exp\frac{-ix_1}{h'} - \exp\frac{-ix_2}{h'}\right)$$

$$x_1 \leqslant x \leqslant x_2$$

$$\Delta h = \frac{\varepsilon_0 h'}{ki^2}\left[1 - \exp\frac{-i(x_2 - x)}{h'} + \frac{i(x - x_1)}{h'} + \exp\frac{-ix_2}{h'} - \exp\frac{-ix_1}{h'}\right]$$

$$x \geqslant x_2$$

$$\Delta h = \frac{\varepsilon_0}{2ki}(x_2 - x_1) \operatorname{erfc}\frac{x - 2\alpha\tau}{2\sqrt{\tau}} + \frac{\varepsilon_0 h'}{ki^2}\left(\exp\frac{-ix_2}{h'} - \exp - \frac{ix_1}{h}\right)$$

b) $x_1, x_2 < 0,\ x_3 = -x_1 > 0,\ x_4 = -x_2 > 0$

$$\begin{aligned} &-\infty < x \leqslant x_1, \quad \Delta h = {}^1/_2 w_0 [I_{1,2} - I_{1,1} + \exp(2\alpha x)(J_{1,4} - J_{1,3})] \\ &x_1 \leqslant x \leqslant x_2, \quad \Delta h = {}^1/_2 w_0 [I_{1,2} - I_{2,1} + \exp(2\alpha x)(J_{1,4} - J_{1,3})] \\ &x_2 \leqslant x \leqslant 0, \quad \Delta h = \frac{w_0}{2}[I_{2,2} - I_{2,1} + \exp(2\alpha x)(J_{1,4} - J_{1,3})] \end{aligned}$$

As $t \to \infty$, we obtain expressions for the limiting value of Δh:

$$-\infty < x \leqslant x_1$$

$$\Delta h = \frac{\varepsilon_0 h'}{ki^2}\exp\frac{ix}{h'}\left[\exp\frac{-ix_1}{h'} - \exp\frac{-ix_2}{h'} - \frac{i(x_2 - x_1)}{h'}\right]$$

$$x_1 \leqslant x \leqslant x_2$$

$$\Delta h = \frac{\varepsilon_0 h'}{ki^2}\left[1 - \exp\frac{-i(x_2 - x)}{h'} - \frac{i(x_1 - x)}{h'}\right] - \frac{\varepsilon_0(x_2 - x_1)}{ki}\exp\frac{ix}{h'}$$

$$x_2 \leqslant x \leqslant 0$$

$$\Delta h = \frac{\varepsilon_0 h'}{ki}(x_2 - x_1)\left(1 - \exp\frac{ix}{h'}\right)$$

In view of the linearity of the starting equation, using the solutions obtained, we can determine Δh when the infiltration on a strip is specified as a piece-wise-constant function of time, and, moreover, take into account the effect on the flow of the infiltration from several infiltration strips.

LITERATURE CITED

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